Math 335: Differential Geometry
Homework 7 (due November 6)

1. Consider $\mathbb{R}^2$ with the usual dot product.
   (a) Suppose that $A$ is a $2 \times 2$ matrix. Show that if $A$ is symmetric, i.e. $A = A^T$, then the linear transformation of $\mathbb{R}^2$ defined by $A$ is self-adjoint. (Hint: confirm first for yourself that if we consider vectors $x$ and $y$ as column vectors, then $x \cdot y = x^T y$.)
   (b) Confirm by direct computation that
   $$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$
   has an orthonormal basis of eigenvectors.

2. Recall that a point $\bar{p}$ in a surface $S$ is call *parabolic* if $dN_{\bar{p}}$ has one eigenvalue equal to 0, and it is called *planar* if $dN_{\bar{p}}$ has both of its eigenvalues equal to 0. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

3. Prove that the image $N \circ \bar{\alpha}$ by the Gauss map $N : S \to S^2$ of a parametrized regular curve $\bar{\alpha} : I \to S$ which contains no planar or parabolic points is a parametrized regular curve on the sphere $S^2$.

4. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $\bar{p} \in S$, is constant.

5. Recall that a point $\bar{p}$ in a surface $S$ is called *elliptic* if both of the eigenvalues of $dN_{\bar{p}}$ have the same sign. Let $C \subset S$ be a regular curve on a surface $S$ all of whose points are elliptic. Show that the curvature $\kappa$ of $C$ at $\bar{p}$ satisfies
   $$\kappa \geq \min(|\kappa_1|, |\kappa_2|)$$
   where $\kappa_1$ and $\kappa_2$ are the principal curvatures of $S$ at $\bar{p}$.

6. Explain why a surface that is compact (i.e. closed and bounded in $\mathbb{R}^3$) has an elliptic point. Hint: to be bounded means that the surface can be completely contained in a sphere of finite radius.