Math 335: Differential Geometry
Homework 6 (due October 30)

1. Let $f : S \to \mathbb{R}$ be given by $f(\bar{p}) = |p - p_0|^2$, where $\bar{p} \in S$ and $\bar{p}_0$ is a fixed point of $\mathbb{R}^3$ not in S. Show that $df_{\bar{p}}(\bar{w}) = 2\bar{w} \cdot (\bar{p} - \bar{p}_0)$, for all $\bar{w} \in T_{\bar{p}}S$.

2. (Gradient on surfaces.) The *gradient* of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map $\mathrm{grad} f : S \to \mathbb{R}^3$ which assigns to each point $\bar{p} \in S$ a vector $\mathrm{grad} f(\bar{p}) \in T_{\bar{p}}S \subset \mathbb{R}^3$ such that
$$\mathrm{grad} f(\bar{p}) \cdot v = df_{\bar{p}}(\bar{v}) \quad \text{for all } \bar{v} \in T_{\bar{p}}S.$$
Show that if E, F, and G are the coefficients of the first fundamental form in a parametrization $\bar{x} : U \subset \mathbb{R}^2 \to S$, then $\mathrm{grad} f$ on $\bar{x}(U)$ is given by
$$\mathrm{grad} f = \frac{f_u G - f_v F}{EG - F^2}\bar{x}_u + \frac{f_v E - f_u F}{EG - F^2}\bar{x}_v,$$
where f_u denotes $\frac{\partial(f \circ \bar{x})}{\partial u}$ and f_v denotes $\frac{\partial(f \circ \bar{x})}{\partial v}$.
(Note that if $S = \mathbb{R}^2$, with chart $\bar{x}(x, y) = (x, y, 0)$, then $\mathrm{grad} f = f_x\bar{e}_1 + f_y\bar{e}_2$, where $\{\bar{e}_1, \bar{e}_2\}$ is the standard basis. Thus the definition agrees with the definition of gradient that we learned in multivariable calculus.)

3. (Chain Rule.) Show that if $\phi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are differentiable maps and $\bar{p} \in S_1$, then
$$d(\psi \circ \phi)_{\bar{p}} = d\psi_{\phi(\bar{p})} \circ d\phi_{\bar{p}}.$$

4. Show that a diffeomorphism $\phi : S_1 \to S_2$ is an isometry if and only if the arc length of any parametrized curve in S is equal to the arc length of the image of the curve by ϕ.

5. A map $G : \mathbb{R}^3 \to \mathbb{R}^3$ is a *distance-preserving* map if
$$|G(\bar{p}) - G(\bar{q})| = |\bar{p} - \bar{q}| \text{ for all } \bar{p}, \bar{q} \in \mathbb{R}^3.$$
Give an example to show that there are local isometries $\phi : S_1 \to S_2$ which cannot be extended into distance-preserving maps $G : \mathbb{R}^3 \to \mathbb{R}^3$.

6. Let S_1, S_2, and S_3 be regular surfaces. Prove that

   (a) If $\phi : S_1 \to S_2$ is an isometry, then $\phi^{-1} : S_2 \to S_1$ is also an isometry.

   (b) If $\phi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are isometries, then $\psi \circ \phi : S_1 \to S_3$ is an isometry.

   For those who have seen abstract algebra: this implies that the set of isometries from a regular surface S to itself forms a group, called the *group of isometries of* S. Here's one way in which abstract algebra shows up in other fields of math!