Math 335: Differential Geometry
Homework 4 (due October 9)

1. Show that the normal line to any point on a parametrized surface given by

$$\bar{x}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)), \qquad f(u) \neq 0, g'(u) \neq 0,$$

passes through the z-axis.

2. Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x,y,z) = 0$, where 0 is a regular value of f, is

$$f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(z - z_0) = 0.$$

3. Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(a, b, 0)$ that lie on the surface and show that they are all parallel to the z-axis. (Note: this surface looks like a nuclear power plant, centered around the z-axis.)

4. If a coordinate neighborhood of a regular surface can be parametrized in the form

$$\bar{x}(u,v) = \bar{\alpha}_1(u) + \bar{\alpha}_2(v)$$

where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are regular parametrized curves, show that the tangent planes along a fixed coordinate curve of this neighborhood are all parallel to a line.

5. Let P be the xy-plane in $\mathbb{R}^3$ and let $\bar{x} : U \to P$ be the parametrization of P given by $\bar{x}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$ (i.e. parametrization by polar coordinates) where $U = \{(r,\theta) \in \mathbb{R}^2 | r > 0, 0 < \theta < 2\pi\}$. Compute the local functions of the first fundamental form (i.e. E, F, and G) of P in this parametrization. Is this parametrization an orthogonal parametrization? Draw a few coordinate curves so that you can confirm intuitively that your answer is correct.

6. Show that a surface of revolution can always be parametrized so that

$$E = E(v) \qquad F = 0 \qquad G = 1.$$

(You can suppose that the axis of revolution is the z-axis.)

7. Suppose that $(\bar{x}, U)$ is a coordinate chart. Show that the lengths of the opposite sides of any quadrilateral formed by coordinate curves are equal if and only if

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0.$$