Math 335: Differential Geometry
Homework 3 (due October 2)

1. Show that the cylinder $\{(x,y,z) \in \mathbb{R}^3 | x^2+y^2=1\}$ is a regular surface and find coordinate charts that cover it. Show that your maps satisfy the definition of being a chart.

2. Is the set $\{(x,y,z) \in \mathbb{R}^3 | z=0 \text{ and } x^2+y^2 \leq 1\}$ a regular surface? Is the set $\{(x,y,z) \in \mathbb{R}^3 | z=0 \text{ and } x^2+y^2 < 1\}$ a regular surface?

3. Let $P = \{(x,y,z) \in \mathbb{R}^3 | x=y\}$ (a plane) and let $\bar{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by
$$\bar{x}(u,v) = (u+v, u+v, uv),$$
where $U = \{(u,v) \in \mathbb{R}^2 | u > v\}$. Clearly, $\bar{x}(U) \subset P$. Is $\bar{x}$ a parametrization of P?

4. Let $\bar{x}(u,v)$ be a parametrization of a regular surface S. Show that $d\bar{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one if and only if
$$\frac{\partial \bar{x}}{\partial u} \times \frac{\partial \bar{x}}{\partial v} \neq 0.$$

5. Let C be a figure 8 in the xy-plane and let S be the cylindrical surface over C, that is,
$$S = \{(x,y,z) \in \mathbb{R}^3 | (x,y) \in C\},$$
(so S is a figure 8 "tube"). Is S a regular surface? Please explain.

6. One way to define a system of coordinates for the sphere S^2, given by $x^2+y^2+(z-1)^2=1$, is to consider the so-called *stereographic projection* $\pi : S^2 \backslash \{N\} \to \mathbb{R}^2$ which carries a point $p=(x,y,z)$ of the sphere S^2 minus the north pole $N=(0,0,2)$ onto the intersection of the xy-plane with the straight line which connects N to p. Let $(u,v) = \pi(x,y,z)$, where $(x,y,z) \in S^2 \backslash \{N\}$ and $(u,v) \in xy$-plane.

   (a) Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by
$$\pi^{-1}(u,v) = \begin{cases} x = \frac{4u}{u^2+v^2+4}, \\ y = \frac{4v}{u^2+v^2+4}, \\ z = \frac{2(u^2+v^2)}{u^2+v^2+4}. \end{cases}$$

(b) Explain how it is possible, using stereographic projection, to cover the sphere with two coordinate charts.

7. Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value of f and yet that $f^{-1}(0)$ is a regular surface.

8. We have proven, by giving a coordinate chart, that the graph of a function $f : U \subset \mathbb{R}^2 \to R$ is a regular surface. Give an alternate proof of this fact by showing that we can think of the graph as the inverse image of a regular value for some function $h : V \subset \mathbb{R}^3 \to \mathbb{R}$. (How should you construct h, based on f?)