Math 335: Differential Geometry
Homework 2 (due September 25)

1. Consider the curve $\bar{\beta}(s) = (\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s)$. Determine the Serret-Frenet apparatus (i.e. $\bar{t}(s), \bar{n}(s), \bar{b}(s), \kappa(s), \tau(s)$) for $\bar{\beta}$. Based on what you find, what can you say about the nature of this curve?

2. If $\bar{\alpha}(s)$ is a unit-speed curve with nonzero curvature, find a vector $w(s)$ such that:
$$\bar{t}'(s) = w(s) \times \bar{t}(s), \qquad \bar{n}'(s) = w(s) \times \bar{n}(s), \qquad \bar{b}'(s) = w(s) \times \bar{b}(s)$$
for all s.

3. A normal line at s to a unit-speed curve in $\mathbb{R}^3$ is the line determined by the normal vector $\bar{n}(s)$. Assume that all normal lines to a unit-speed parametrized curve $\bar{\alpha} : I \to \mathbb{R}^3$ pass through a fixed point. Prove that the trace of the curve is contained in a circle.

4. Let $\bar{\alpha} : I \to \mathbb{R}^2$ be a unit-speed curve. Let $\{\bar{t}(s), \bar{n}(s)\}$ be the positively oriented tangent-normal basis to $\bar{\alpha}(s)$ (i.e. $\bar{n}(s)$ is $90°$ counter-clockwise from $\bar{t}(s)$) and let $\kappa(s)$ be the *signed* curvature at $\alpha(s)$. Assume that $\kappa(s) \neq 0$ for all $s \in I$. In this situation, the curve
$$\bar{\beta}(s) = \bar{\alpha}(s) + \frac{1}{\kappa(s)}\bar{n}(s), \quad s \in I$$
is called the *evolute* of $\bar{\alpha}$.

   Show that the tangent line at s of the evolute of $\bar{\alpha}$ is the normal line to $\bar{\alpha}$ at s.

5. Show that the knowledge of the vector function $\bar{b}(s)$ (the binormal vector) of a curve $\bar{\alpha}$, with nonzero torsion everywhere, determines the curvature $\kappa(s)$ and the absolute value of the torsion $\tau(s)$ of $\bar{\alpha}$.