Math 335: Differential Geometry
Homework 10 (due in class on December 8)

1. Let S be a sphere of radius 5, centered at the origin in $\mathbb{R}^3$ and oriented outward. Suppose that C is the horizontal cross section of S at height 4, parametrized by $\bar{\alpha}(t) = (3\cos t, 3\sin t, 4)$. Consider the point $\bar{p} = \bar{\alpha}(0) = (3, 0, 4)$.
   (a) Compute the basis $\{\bar{t}, \bar{N} \times \bar{t}, \bar{N}\}$ at $\bar{p}$. Here $\bar{t}$ denotes the unit tangent to C in the direction of $\bar{\alpha}$ and $\bar{N}$ denotes the unit normal vector to the sphere. Draw a picture that indicates the curve and this basis.
   (b) Compute $\frac{D\bar{\alpha}'(0)}{dt}$. Is $\bar{\alpha}$ geodesic at $\bar{p}$?
   (c) Reparametrize $\bar{\alpha}$ with respect to arc length.
   (d) Compute κ, κ_g, and κ_n of $\bar{\alpha}$ at $\bar{p}$ directly. Confirm that $\kappa^2 = \kappa_g^2 + \kappa_n^2$.

2. Recall that a regular curve C in a surface S is called a *line of curvature* if at each point $\bar{p} \in C$, the tangent to C is a principal direction at $\bar{p}$. Show that if a curve $C \subset S$ is both a line of curvature and a geodesic, then C is a plane curve.

3. Give an example of a line of curvature which is a plane curve and not a geodesic.

4. Prove that a curve $C \subset S$ is both an asymptotic curve and a geodesic if and only if C is a straight line.

5. Show that there are infinitely many geodesics joining any two points (x, y, z) and (x', y', z') on the unit cylinder.

6. Let S be an oriented regular surface and let $\bar{\alpha} : I \to S$ be a curve parametrized by arc length. At the point $\bar{p} = \bar{\alpha}(s)$ consider the three unit vectors $\bar{t} = \bar{\alpha}'(s)$, $\bar{N}$ the unit normal vector to S at $\bar{\alpha}(s)$, and $\bar{N} \times \bar{t}$. Show that

$$\begin{aligned}
\frac{d\bar{t}}{ds} &= 0\bar{t} + a(s)(\bar{N} \times \bar{t}) + b(s)\bar{N} \\
\frac{d(\bar{N} \times \bar{t})}{ds} &= -a(s)\bar{t} + 0(\bar{N} \times \bar{t}) - c(s)\bar{N} \\
\frac{d\bar{N}}{ds} &= -b(s)\bar{t} + c(s)(\bar{N} \times \bar{t}) + 0\bar{N}.
\end{aligned}$$

For some functions $a(s)$, $b(s)$, and $c(s)$. The above formulas are the analogues of the Serret-Frenet formulas for the orthonormal moving frame $\{\bar{t}(s), \bar{N}_{\bar{\alpha}(s)} \times \bar{t}(s), \bar{N}_{\bar{\alpha}(s)}\}$. To establish the geometric meaning of the coefficients, prove that:

(a) $c(s) = (\frac{d\bar{N}}{ds}) \cdot (\bar{N} \times \bar{t})$; conclude from this that $\bar{\alpha}(I) \subset S$ is a line of curvature if and only if $c(s) = 0$ for all s. ($-c$ is call the *geodesic torsion* of $\bar{\alpha}$.)

(b) $b(s)$ is the normal curvature of $\bar{\alpha}(I) \subset S$ at $\bar{p}$.

(c) $a(s)$ is a geodesic curvature of $\bar{\alpha}(I) \subset S$ at $\bar{p}$.

7. (a) Create two triangulations of the sphere, distinct from the one described in the video/notes. Compute the Euler characteristic of the sphere two times, once from each of your triangulations. Do you get the same thing both times?

   (b) Create a triangulation of the torus. What is the Euler characteristic of the torus?

8. (For the last day of class, not to be turned in.)

   The general version of the global Gauss-Bonnet theorem states that if $R \subset S$ is a regular region of an oriented surface bounded by a simple, closed, piecewise regular curve, then

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)\, ds + \iint_R K\, dA + \sum_{j=1}^{p} \theta_j = 2\pi\chi(R)$$

   where C_i, $i = 1, \ldots, n$, are the segments of the boundary of R and θ_j, $j = 1, \ldots p$, are the exterior angles of the boundary of R.

   A *simply-connected* region in a surface S is a region that is homeomorphic to a disk (it has no holes in it and is bounded by a closed, simple, piecewise-differentiable curve). An *n-gon* on a surface S is a piecewise regular curve α with n vertices whose segments are geodesics and which bounds a simply connected region. Let S be a surface with negative Gaussian curvature everywhere. Show that there are no n-gons in S for $n = 0, 1, 2$. Please explain.