Math 335: Differential Geometry
Homework 1 (due September 18)

1. If $\bar{\alpha}(t) : \mathbb{R} \to \mathbb{R}^2$ is such that $\bar{\alpha}''(t) = \bar{0}$ for all t, what can be said of $\bar{\alpha}(t)$?

2. Find a parametrized curve $\bar{\alpha}(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\bar{\alpha}(t)$ runs clockwise around the circle with $\bar{\alpha}(0) = (0, 1)$.

3. Show that for vectors $\bar{u}(t)$ and $\bar{v}(t)$ in $\mathbb{R}^n$, $(\bar{u}(t) \cdot \bar{v}(t))' = \bar{u}'(t) \cdot \bar{v}(t) + \bar{u}(t) \cdot \bar{v}'(t)$.

4. Let $\bar{\alpha}(t)$ be a parametrized curve that does not pass through the origin. If $\bar{\alpha}(t_0)$ is the point of the trace of $\bar{\alpha}$ closest to the origin and $\bar{\alpha}'(t_0) \neq 0$, show that the position vector $\bar{\alpha}(t_0)$ is orthogonal to $\bar{\alpha}'(t_0)$.

5. Show that the tangent lines to the regular parametrized curve $\bar{\alpha}(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0, z = x$.

6. Let $\bar{\alpha}(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$, $t \in \mathbb{R}$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

   (a) Show that as $t \to +\infty$, $\bar{\alpha}(t)$ approaches the origin, spiraling around it (because of this, the trace of $\bar{\alpha}$ is called the *logarithmic spiral*). Draw a sketch.

   (b) Show that $\bar{\alpha}'(t) \to (0, 0)$ as $t \to +\infty$ and that
   $$\lim_{t \to +\infty} \int_{t_0}^{t} |\bar{\alpha}'(t)|\, dt$$
   is finite; that is, $\bar{\alpha}$ has finite arc length in $[t_0, \infty)$.

7. Let $\bar{\alpha} : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\bar{\alpha}(a) = \bar{p}$, $\bar{\alpha}(b) = \bar{q}$.

   (a) Show that, for any constant vector $\bar{v}$ with $|v| = 1$,
   $$(\bar{q} - \bar{p}) \cdot v = \int_a^b \bar{\alpha}'(t) \cdot \bar{v}\, dt \leq \int_a^b |\bar{\alpha}'(t)|\, dt.$$
   (Hint: the first equality is the harder of the two. You will need the Fundamental Theorem of Calculus.)

(b) Set

$$v = \frac{\bar{q} - \bar{p}}{|\bar{q} - \bar{p}|}$$

and show that

$$|\bar{\alpha}(b) - \bar{\alpha}(a)| \leq \int_a^b |\bar{\alpha}'(t)|\, dt;$$

that is, the curve of shortest length from $\bar{\alpha}(a)$ to $\bar{\alpha}(b)$ is the straight line joining these points.